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Dynamic Stability of Doubly Curved Orthotropic Shallow Shells Under Impact

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A methodology for the stability analysis of doubly curved orthotropic shells with simply supported boundary condition and under impact load from the viewpoint of nonlinear dynamics is studied. The nonlinear governing differential equations are derived based on a Donnell-type shallow shell theory, and the displacement is projected onto the space spanned by the eigenfunction of the linear operator of the motion equation. To analyze the influence of each single mode pair on the response to impact loading, only one term composed of two half-waves is used in developing the governing equation, whereas the first mode pair, which is close to the membrane state, is used in the numerical examples. For the first case analyzed, it turns out that two centers and one saddle will occur if damping is ignored, whereas the two centers become two focuses and the saddle point remains if damping is considered. The nonlinear behavior also has been investigated by neglecting the influence of inertia and damping, and the results show that two saddle-node bifurcations will occur under certain conditions.

Nomenclature

a, b, h	=	length, width and thickness of panel
d	=	damping per area
E_1, E_2	=	modulus in longitudinal and transverse directions
G_{12}	=	shear modulus
g	=	gravity
k_x, k_y	=	the curvatures in x and y directions
$\bar{k}$	=	load level
M_1, M_2, H	=	moments per unit length
N_1, N_2, S	=	forces per unit length
T	=	impact period
u, v, w	=	middle surface displacement components
w_{nm}	=	unknown coefficients for function w
γ_{12}, γ_{21}	=	Poisson's ratio
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}, k_1, k_2, \tau$	=	panel strain components
ρ	=	mass density of panel
φ	=	stress function

Introduction

DOUBLY curved orthotropic shells under impact load are encountered frequently in practice, and the dynamic stability (buckling and post-buckling) behavior of such a thin-walled structure can be one of the important design specifications. To achieve a design with higher impact stability levels, the dynamic stability behavior should be understood and predicted by investigating the dynamic stability mechanisms and their dependence on the system parameters.

Strictly speaking, from the viewpoint of nonlinear dynamics, buckling is a static bifurcation. That means there will exist multiple equilibrium positions for the system at a critical parameter value. Therefore, one needs to analyze the stability of the equilibrium positions for the prediction of buckling. For the system studied in this paper, to investigate the influence of the impact load on the buckling behavior, global stability analysis should be applied to determine the attraction domains of the equilibrium positions. An important and well-known aspect of nonlinear dynamics is the sensitive dependence of the solution on the perturbation. Such perturbations include small variations in initial and boundary values, as well as numerical errors if a numerical computation method is adopted. In most finite element procedures reported in literature, a thin-plate model with large deformations is used to approach the deformation of thin-walled curved panels.[1] Hence, for a doubly curved shallow panel, in every element the influence of curvature on the strain for a doubly curved shallow panel has been neglected. If this approach is used, improved results can be obtained if the mesh is refined to a sufficient degree. However, the nonlinear behavior of some types of thin-walled doubly curved panels under large deformation is very sensitive to their curvatures. Because of this and to investigate the mechanism of dynamic buckling, a semi-analytical method is applied to investigate doubly curved shallow panels with a rectangular boundary and simply supported boundary condition subjected to impact load.

Many researchers have focused their studies on the buckling behavior of thin-walled structures, especially on static buckling.[2–4] However, some of those structures can be modeled to a sufficient accuracy by a static system, whereas others cannot. Ette[5] investigated the dynamic buckling of an imperfect spherical shell based on a perturbation method. Riks et al.[6] have presented a method for simulating the transient buckling numerically based on the combination of a classical path-following method with a transient integration method. To model the dynamic response and establish postbuckling/post-snap-through equilibrium of discrete structures, Hilburger et al.[7] have developed a transient analysis that incorporates dynamic effects during the transition from one equilibrium state to another. Several semi-analytical methods based on a single mode pair have been presented for the analysis of the nonlinear free vibrations of curved panels with large deformation,[8–11] which show that the large-amplitude oscillations of panels are dominated by the lower modes.

This paper focuses on a methodology for the analysis of the stability of doubly curved orthotropic shells with simply supported

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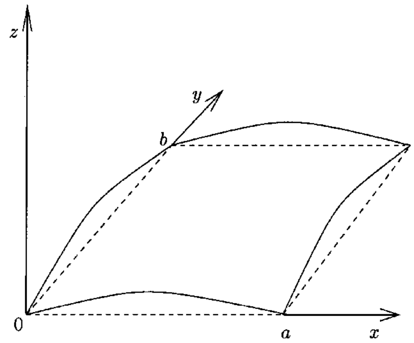

Fig. 1 Doubly curved orthotropic panels.

boundary condition and under impact load, from the nonlinear dynamics viewpoint. In the nonlinear dynamic analysis, only one mode pair has been considered. From the cited literature, one mode pair can be expected to yield a reasonable first approximation. The influence of higher modes on the dynamics and the interaction between them are currently under investigation.

Governing Equations

Figure 1 shows the doubly curved shell with simply supported boundary conditions and subjected impact load. In the following analysis, the impact load is approximated by the model of the half-sinusoidal shock pulse velocity change.

The Donnell-type strain–displacements relations for a shallow curved shell are

$$\varepsilon_x = u_{,x} + k_x w + \tfrac{1}{2}(w_{,x})^2, \qquad \varepsilon_y = v_{,y} + k_y w + \tfrac{1}{2}(w_{,y})^2$$

$$\varepsilon_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y}$$

$$k_1 = -w_{,xx}, \qquad k_2 = -w_{,yy}, \qquad \tau = -2w_{,xy} \tag{1}$$

The corresponding stress–strain relationships can be expressed as

$$N_1 = (E_1 h/B)(\varepsilon_x + \gamma_{21}\varepsilon_y), \qquad N_2 = (E_2 h/B)(\varepsilon_y + \gamma_{12}\varepsilon_x)$$

$$S = G_{12} h \varepsilon_{xy} \tag{2}$$

$$M_1 = \left(E_1 h^3 \big/ 12B\right)(k_1 + \gamma_{21} k_2)$$

$$M_2 = \left(E_2 h^3 \big/ 12B\right)(k_2 + \gamma_{12} k_1), \qquad H = \left(G_{12} h^3 \big/ 12\right)\tau \tag{3}$$

where $B = 1 - \gamma_{12}\gamma_{21}$.

The simply supported boundary conditions on $x = 0, a$ are

$$w = N_1 = M_1 = 0$$

and on $y = 0, b$ are

$$w = N_2 = M_2 = 0$$

According to shallow shell theory, a stress function φ can be introduced that satisfies the following equations:

$$N_1 = h\varphi_{,yy}, \qquad N_2 = h\varphi_{,xx}, \qquad S = -h\varphi_{,xy} \tag{4}$$

Based on the stress–strain relationships in terms of the stress function φ and the displacement w, the compatibility equation can be derived to be

$$D_7 h\varphi_{,xxxx} + D_{11} h\varphi_{,xxyy} + D_5 h\varphi_{,yyyy} = k_x w_{,yy} + k_y w_{,xx} + (w_{,xy})^2 - w_{,xx} w_{,yy} \tag{5}$$

Furthermore, using Hamilton's principle, the equation of motion of the system under impact can be shown in terms of φ and displacement w to be

$$D_1 w_{,xxxx} - D_{10} w_{,xxyy} + D_2 w_{,yyyy} + h\varphi_{,yy}(k_x - w_{,xx}) + h\varphi_{,xx}(k_y - w_{,yy}) + 2h\varphi_{,xy} w_{,xy} - q = 0 \tag{6}$$

The parameters in Eqs. (5) and (6) are defined to be

$$q = \rho h q^* - \rho h \ddot{w} - d\dot{w}$$

$$q^* = \bar{k} g \sin(\omega t) \qquad \text{if} \qquad 0 \le t < T$$

$$\omega = \pi/T, \qquad \bar{k} \in Z^+$$

$$q^* = 0 \qquad \text{if} \qquad t \ge T$$

$$B_1 = E_1 h/B, \qquad B_2 = E_2 h/B, \qquad B_3 = G_{12} h$$

$$D_1 = E_1 h^3 \big/ 12B, \qquad D_2 = E_2 h^3 \big/ 12B, \qquad D_3 = G_{12} h^3 \big/ 12$$

$$D_4 = 1/1 - \gamma_{12}\gamma_{21}, \qquad D_5 = D_4/B_1, \qquad D_6 = (\gamma_{21}/B_2) D_4$$

$$D_7 = D_4/B_2, \qquad D_8 = (\gamma_{12}/B_1) D_4, \qquad D_9 = 1/B_3$$

$$D_{10} = -D_1\gamma_{21} - 4D_3 - D_2\gamma_{12}, \qquad D_{11} = -D_6 - D_8 + D_9$$

Note that an overdot indicates the partial derivative with respect to time t, whereas T is the impact pulse duration. Let $\bar{V}$ be a Hilbert space, the scalar product and the norm on $\bar{V}$ are denoted by $(\cdot,\cdot)$ and $|\cdot,\cdot|$, respectively. Here, $w \in L^2(0, TT; \bar{V})$, where $[0, TT]$ is the interval of time. For reasons of convenience, the linear operator of the equation of motion together with the boundary conditions can be defined as $Aw = D_1 w_{,xxxx} - D_{10} w_{,xxyy} + D_2 w_{,yyyy}$. Then $\{\sin(n\alpha x)\sin(m\beta y),\ n, m = 1, 2, 3, \cdots + \infty\}$ constitute the set of eigenfunctions that form an orthogonal basis of the space $\bar{V}$.

Because of the nonlinearity of Eqs. (5) and (6), an approximation to the solution of the infinite-dimensional evolution equation is constructed as a summation of spatially doubly sinusoidal and temporally synchronized displacement functions that satisfy the simply supported boundary condition. In other words, the solution $w(x, y, t)$ is projected onto the space spanned by $\{\sin(n\alpha x)\sin(m\beta y),\ n, m = 1, 2, 3, \cdots + \infty\}$, which is complete in the space $\bar{V}$, that is,

$$w(x, y, t) = -\sum\sum w_{nm}(t)\sin(n\alpha x)\sin(m\beta y) \tag{7}$$

where $\alpha = \pi/a$ and $\beta = \pi/b$. Here, n and m are referred to as the half-wave numbers in the x and y directions, respectively. It is obvious that axisymmetric modes correspond to odd half-wave numbers and asymmetric modes correspond to even half-wave numbers.

To analyze the influence of each single mode on the response to the impact load, only one term, composed of two half-waves, has been retained, that is,

$$w(x, y, t) \approx -w_{nm}(t)\sin(n\alpha x)\sin(m\beta y) \tag{8}$$

The stress function φ can be obtained by substituting Eq. (8) into Eq. (5) and satisfying the continuous boundary conditions on an average:

$$\varphi_{nm} = d_1 (w_{nm})^2 \cos(2n\alpha x) + d_2 (w_{nm})^2 \cos(2m\beta y) + d_3 w_{nm} \sin(n\alpha x)\sin(m\beta y) \tag{9}$$

where

$$d_1 = \frac{m^2\beta^2}{32 D_7 h n^2 \alpha^2}, \qquad d_2 = \frac{n^2\alpha^2}{32 D_5 h m^2 \beta^2}$$

$$d_3 = \frac{k_x m^2\beta^2 + k_y n^2\alpha^2}{D_5 h m^4\beta^4 + D_7 h n^4 \alpha^4 + D_{11} h n^2\alpha^2 m^2\beta^2}$$

A Galerkin procedure is used to construct an approximate solution to Eq. (6), that is, the residual error $L(x, y, w)$ is required to be orthogonal to the coordinate functions w_{nm}, in an averaged way:

$$\int_0^b \int_0^a L(x, y, w)\delta w \, dx \, dy = 0 \tag{10}$$

where

$$L(x, y, w) = D_1 w_{,xxxx} - D_{10} w_{,xxyy} + D_2 w_{,yyyy} + h\varphi_{,yy}(k_x - w_{,xx}) + h\varphi_{,xx}(k_y - w_{,yy}) + 2h\varphi_{,xy} w_{,xy} - q$$

It follows that quadratic and cubic nonlinearities in terms of w_{nm} will be included in the residual error $L(x, y, w)$. Indeed, substituting Eqs. (8) and (9) into $L(x, y, w)$ yields

$$L(x, y, w_{nm}) = -\big(D_{23}\ddot{w}_{nm} + D_{27}\dot{w}_{nm} + D_{24}(w_{nm})^3 + D_{25}(w_{nm})^2 + D_{26}w_{nm} + \rho h q^*\big) \tag{11}$$

where

$$D_{12} = \big(D_1 n^4\alpha^4 + D_2 m^4\beta^4 - D_{10} n^2\alpha^2 m^2\beta^2\big)\sin(n\alpha x)\sin(m\beta y)$$

$$D_{13} = 4hd_2 m^2\beta^2 k_x \cos(2m\beta y)$$

$$D_{14} = -4hd_2 n^2\alpha^2 m^2\beta^2 \cos(2m\beta y)\sin(n\alpha x)\sin(m\beta y)$$

$$D_{15} = hd_3 m^2\beta^2 k_x \sin(n\alpha x)\sin(m\beta y)$$

$$D_{16} = D_{20} = -hd_3 n^2\alpha^2 m^2\beta^2 \sin^2(n\alpha x)\sin^2(m\beta y)$$

$$D_{17} = 4hd_1 n^2\alpha^2 k_y \cos(2n\alpha x)$$

$$D_{18} = -4hd_1 n^2\alpha^2 m^2\beta^2 \cos(2n\alpha x)\sin(n\alpha x)\sin(m\beta y)$$

$$D_{19} = hd_3 n^2\alpha^2 k_y \sin(n\alpha x)\sin(m\beta y)$$

$$D_{21} = 2hd_3 n^2\alpha^2 m^2\beta^2 \cos^2(n\alpha x)\cos^2(m\beta y)$$

$$D_{23} = \rho h \sin(n\alpha x)\sin(m\beta y), \qquad D_{24} = D_{14} + D_{18}$$

$$D_{25} = D_{13} + D_{16} + D_{17} + D_{20} + D_{21}$$

$$D_{26} = D_{12} + D_{15} + D_{19}, \qquad D_{27} = d\sin(n\alpha x)\sin(m\beta y)$$

When the integration based on Eq. (10) is carried out, a single second-order nonlinear ordinary differential equation of motion with quadratic and cubic nonlinearities is obtained in the form

$$D_{53}\ddot{w}_{nm} + D_{57}\dot{w}_{nm} + D_{54}(w_{nm})^3 + D_{55}(w_{nm})^2 + D_{56}w_{nm} + (4\rho h/n\alpha m\beta)q^* = 0 \tag{12}$$

where

$$D_{53} = \rho hab/4, \qquad D_{54} = (hn^2\alpha^2 m^2\beta^2 ab/2)(d_1 + d_2)$$

$$D_{55} = -(8h/3n\alpha m\beta)\big(2d_2 k_x m^2\beta^2 + 2d_1 k_y n^2\alpha^2 + d_3 n^2\alpha^2 m^2\beta^2\big)$$

when neither n nor m is even, whereas

$$D_{55} = 0$$

when either n or m is even, and

$$D_{56} = (ab/4)\big(D_1 n^4\alpha^4 + D_2 m^4\beta^4 - D_{10} n^2\alpha^2 m^2\beta^2 + hd_3 k_x m^2\beta^2 + hd_3 k_y n^2\alpha^2\big)$$

$$D_{57} = dab/4$$

Stability Analysis and Criterion

Buckling means that the system has multiple equilibrium positions under a certain load. To understand the buckling of the shell subjected to impact load, it is necessary to investigate its equilibrium positions and their stability.

To analyze the nonlinear dynamic behavior, Eq. (12) is rewritten as a set of first-order equations referring to the state space spanned by (w_{nm}, u_{nm}), with $u_{nm} = \dot{w}_{nm}$,

$$\begin{Bmatrix} \dot{w}_{nm} \\ \dot{u}_{nm} \end{Bmatrix} = \boldsymbol{F}(w_{nm}, u_{nm}, t) = \begin{Bmatrix} u_{nm} \\ DD_1 \end{Bmatrix} \tag{13}$$

In Eq. (13) the quantity DD_1 is defined to be

$$DD_1 = -(1/D_{53})\big(D_{57}u_{nm} + D_{54}(w_{nm})^3 + D_{55}(w_{nm})^2 + D_{56}w_{nm} + D_{58}\big)$$

$$D_{58} = 4\rho h q^*/n\alpha m\beta$$

The velocity change Δv after impact can be calculated from $\Delta v = 0.637\bar{k}gT$ based on the half-sinusoidal shock pulse velocity change.

After the preceding transformation, the motion of the structure is clearly described by its position and velocity as functions of time and the initial conditions. Because of the impact and the damping, this system will approach a static state, that is, equilibrium position, after a certain time. In fact, the flow $\phi_t(x)$ described by Eq. (13) is a commutative group with one parameter that satisfies both $\phi_0(x) = x$ and $\phi_{t+s}(x) = \phi_t(\phi_s(x))$ for all $x \in E$, where E is an open subset of $\boldsymbol{R}^2$. Thus, Eq. (13) can be rewritten in the following form:

$$\begin{Bmatrix} \dot{w}_{nm} \\ \dot{u}_{nm} \end{Bmatrix} = \boldsymbol{F}(w_{nm}, u_{nm}) = \begin{Bmatrix} u_{nm} \\ DD_2 \end{Bmatrix} \tag{14}$$

where

$$DD_2 = -(1/D_{53})\big(D_{57}u_{nm} + D_{54}(w_{nm})^3 + D_{55}(w_{nm})^2 + D_{56}w_{nm}\big)$$

As an approximation, the initial velocity and displacement immediately after the impact have been set to $\dot{w}|_{t=0} = -\Delta v$ and $w|_{t=0} = 0$, respectively. Equation (14) can be considered as an equation for free vibrations with certain initial conditions.

From Eq. (14), by definition, equilibrium positions satisfy the equations

$$\boldsymbol{F}(w_{nm}, u_{nm}) = \{0\} \tag{15}$$

which can be rewritten as

$$D_{54}(w_{nm})^3 + D_{55}(w_{nm})^2 + D_{56}w_{nm} = 0 \tag{16}$$

The solutions to Eq. (16) are

$$w_{nm}|_1 = 0, \qquad w_{nm}|_{2,3} = \frac{-D_{55} \pm \sqrt{(D_{55})^2 - 4D_{54}D_{56}}}{2D_{54}} \tag{17}$$

It can be seen that the system will have at least one equilibrium position, namely, $w_{nm}|_1 = 0$, whereas the existence of other two equilibrium positions depends on the expression $\Delta = (D_{55})^2 - 4D_{54}D_{56}$. If $\Delta \geq 0$, the system will have two other equilibrium positions. Otherwise, the system will have only one equilibrium position. Therefore, the algebraic equation $\Delta = 0$ can be looked on as a bifurcation or buckling criterion.

In general, the stability and type of the equilibrium positions can be determined by the magnitude of the eigenvalues of the Jacobian matrix of the vector $\boldsymbol{F}$, that is,

$$\lambda^i_{1,2} = -(D_{57}/D_{53}) \pm D_{59}/2 \tag{18}$$

where

$$D_{59} = \sqrt{(D_{57}/D_{53})^2 - (4/D_{53})DD_3}$$

$$DD_3 = 3D_{54}(w_{nm}|_i)^2 + 2D_{55}w_{nm}|_i + D_{56}$$

Consequently, the equilibrium positions of the autonomous system Eq. (14) can be classified as follows.

1) If both λ_1 and λ_2 are real, and $\lambda_1\lambda_2 > 0, \lambda_1 \neq \lambda_2$, the equilibrium position is called a node.

2) If both λ_1 and λ_2 are real, and $\lambda_1\lambda_2 < 0$, the equilibrium position is called a saddle point.

3) If λ_1 and λ_2 are complex conjugates with nonzero real part, the corresponding equilibrium position is called an unstable focus [$\mathrm{Re}(\lambda_1) > 0$] or a stable focus [$\mathrm{Re}(\lambda_1) < 0$].

4) An equilibrium position whose eigenvalues have zero real part is called a nonhyperbolic equilibrium position. The stability of a nonhyperbolic position cannot be determined from the eigenvalues alone.[12]

For the free vibration of curved shallow panels with large deformations, the axisymmetric part of the assumed deflected shape plays an important role in the nonlinear behavior, and the nonlinear dynamic behavior is dominated by the lower modes.[8–9] Raouf and Palazotto[10] have used a Lindstedt–Poincaré perturbation technique to analyze the influence of higher modes on the nonlinear behavior in free vibration and showed that large-amplitude motions of the curved panels are dominated by the lower modes. They drew a conclusion that the large amplitude, free oscillations cause mainly membrane action and little bending. Indeed, the first mode pair in the eigenfunction defined earlier is close to the membrane state. Therefore, only one term composed of the two first half-waves ($n = m = 1$) has been used to approach the response of the nonlinear free vibration governed by Eq. (14). In other words, only the first two half-waves have been considered as the approximate solution.

For convenience, in Eq. (14) w_{nm} is replaced by w and u_{nm} is replaced by u_1, according to Eq. (8). Moreover, the independent coordinates are written as x_0 and y_0, so that $w(x_0, y_0, t)$ and $u_1(x_0, y_0, t)$. Herein, (x_0, y_0) is an arbitrary point of the shell (not belonging to the boundary). Hence, Eq. (14) is rewritten as

$$\begin{Bmatrix} \dot{w} \\ \dot{u}_1 \end{Bmatrix} = \boldsymbol{F}(w, u_1) = \begin{Bmatrix} u_1 \\ DD_4 \end{Bmatrix} \tag{19}$$

where

$$DD_4 = -(1/D_{53})\big(D_{57}u_1 + D_{64}w^3 + D_{65}w^2 + D_{56}w\big)$$

$$D_{64} = D_{54}\Big/\sin^2(\alpha x_0)\sin^2(\beta y_0)$$

$$D_{65} = -D_{55}\Big/\sin(\alpha x_0)\sin(\beta y_0)$$

As discussed before, the equilibrium positions can be obtained by resolving the algebraic equation (16), which becomes in the modified notation, etc.,

$$D_{64}w^3 + D_{65}w^2 + D_{56}w = 0 \tag{20}$$

The solutions of Eq. (20) are

$$w_1 = 0, \qquad w_{2,3} = \frac{-D_{65} \pm \sqrt{(D_{65})^2 - 4D_{64}D_{56}}}{2D_{64}} \tag{21}$$

The local stability of the equilibrium positions is determined by the eigenvalues of Jacobian matrix of vector $\boldsymbol{F}$, that is,

$$\lambda^i_{1,2} = -(D_{57}/D_{53}) \pm D_{60}/2 \tag{22}$$

where

$$DD_5 = 3D_{64}(w_i)^2 + 2D_{65}w_i + D_{56}$$

$$D_{60} = \sqrt{(D_{57}/D_{53})^2 - (4/D_{53})DD_5}$$

If damping is neglected, a Hamiltonian system can be obtained from Eq. (19). Its phase portrait is determined by the tangent

$$\frac{\mathrm{d}w}{\mathrm{d}u_1} = \frac{-D_{53}u_1}{D_{64}w^3 + D_{65}w^2 + D_{56}w} \tag{23}$$

The first integral of Eq. (23) is

$$-(D_{53}/2)(u_1)^2 = \tfrac{1}{4}D_{64}w^4 + \tfrac{1}{3}D_{65}w^3 + \tfrac{1}{2}D_{56}w^2 + c_1 \tag{24}$$

where c_1 is a constant of the integration associated with initial values. Let the function $V(w, u_1)$ be defined as

$$V(w, u_1) = \tfrac{1}{2}(u_1)^2 + (1/D_{53})\big(\tfrac{1}{4}D_{64}w^4 + \tfrac{1}{3}D_{65}w^3 + \tfrac{1}{2}D_{56}w^2\big) \tag{25}$$

Then $V(w, u_1) \in C^2(E)$. Now, the Hamiltonian system governed by Eq. (19) with $D_{57} = 0$ can be expressed as follows:

$$\dot{w} = \frac{\partial V}{\partial u_1}, \qquad \dot{u}_1 = -\frac{\partial V}{\partial w} \tag{26}$$

This Hamiltonian system completely characterized the flow in the (w, u_1) phase plane. Note that $V(w, u_1)$ can be considered to be a Hamiltonian function. All Hamiltonian systems are conservative in the sense that the Hamiltonian function or the total energy $V(w, u_1)$ remains constant along trajectories of the system, that is, $\mathrm{d}V/\mathrm{d}t = 0$. The trajectories of Eq. (26) lie on the surfaces of $V(w, u_1) = \text{const}$. They are symmetric with respect to the line governed by $u_1 = 0$. These characteristics can easily be proved. Indeed, the total energy for the system without damping is $H(w, u_1) = T(u_1) + U(w)$, where $T(u_1) = \frac{1}{2}(u_1)^2$ accounts for the kinetic energy and $U(w) = (1/D_{53})(\frac{1}{4}D_{64}w^4 + \frac{1}{3}D_{65}w^3 + \frac{1}{2}D_{56}w^2) + c_2$ accounts for the potential energy, where the constant c_2 is associated with initial values.

Notice that the introduction of damping will lead to the breaking of the symmetry, that is, homoclinic bifurcation.

Numerical Examples

Based on the preceding formulation and criteria for the equilibrium positions and their stability behavior, a software program has been developed to analyse the stability behavior of doubly curved orthotropic shells under impact load. The system parameters are as follows: $\rho = 4450\ \mathrm{kg/m^3}$, $E_1 = 3 \times 10^{10}\ \mathrm{N/m^2}$, $E_2 = 4 \times 10^{10}\ \mathrm{N/m^2}$, $G_{12} = 2.6 \times 10^{10}\ \mathrm{N/m^2}$, $h = 0.00022$ m, $\gamma_{12} = 0.15$, $\gamma_{21} = 0.2$, $k_x = 0.4\ \mathrm{m^{-1}}$, $k_y = 0.4\ \mathrm{m^{-1}}$, $d = 200\ \mathrm{Ns/m^3}$, $a = 0.2$ m, $b = 0.2$ m, and $T = 0.035$ s. Note that all of the phase portraits to be shown represent the response at the center of the shell.

Case 1, Hamiltonian System with Three Equilibrium Positions

The Hamiltonian system will be investigated for comparison. For such a system, there exist two centers and one saddle. The stable and unstable manifolds w^s and w^u of the saddle intersect nontransversely, and can be defined separatrices. Figure 2 shows the phase portrait. The equilibrium positions and their stability are listed in

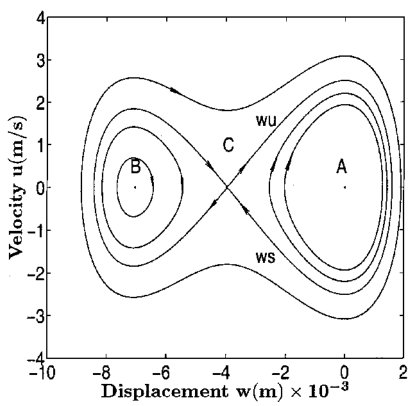

Fig. 2 Phase portrait of system without damping.

Table 1 Equilibrium positions and their stability for the system without damping

Position no.	Position, m	Re(λ_1)	Im(λ_1)	Re(λ_2)	Im(λ_2)
A	0.0	0.0	-0.1291×10^4	0.0	0.1291×10^4
B	-0.7034×10^{-2}	0.0	-0.1135×10^4	0.0	0.1135×10^4
C	-0.3969×10^{-2}	-0.8524×10^3	0.0	0.8524×10^3	0.0

Table 2 Equilibrium positions and their stability for the system with damping

Position no.	Position, m	Re(λ_1)	Im(λ_1)	Re(λ_2)	Im(λ_2)
A	0.0	-0.1022×10^3	-0.1287×10^4	-0.1022×10^3	0.1287×10^4
B	-0.7034×10^{-2}	-0.1022×10^3	-0.1130×10^4	-0.1022×10^3	0.1130×10^4
C	-0.3969×10^{-2}	-0.9607×10^3	0.0	0.7564×10^3	0.0

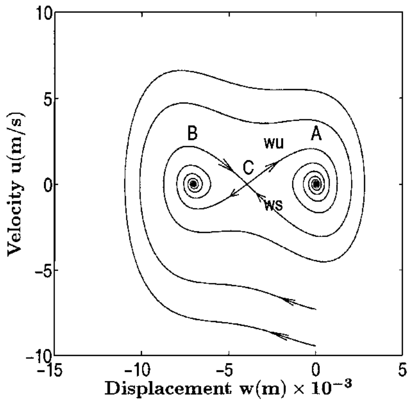

Fig. 3 Phase portrait of system with damping.

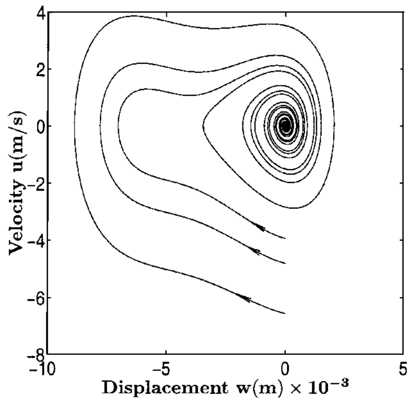

Fig. 4 Deformation vs load.

Table 1. Moreover, it can be seen that the trajectories are symmetric with respect to the line governed by $u_1 = 0$, as expected before.

Case 2, Damped System with Three Equilibrium Positions

If damping is considered, then there exist two stable focuses and one saddle point, which implies that the centers become focuses and the saddle point remains. All of these results are listed in Table 2. Figure 3 shows the corresponding phase portrait. It is obvious that the stable and unstable manifolds w^s and w^u of the saddle C do not intersect each other, whereas the stable manifold of the saddle divides the phase space into two regions. The region to the right of w^s is the basin of attraction of the focus A. This implies that a response starting with initial values inside this region will reach focus A after a certain time. The region to the left of w^s is the basin of attraction of the focus B, which implies that a response starting with initial values inside this region will reach focus B after a certain time, implying that buckling (snap through) occurs. Hsu (see Ref. 13) has given a description of the notation of snap-through instability. In this case, all initial displacements are kept fixed, namely, $w|_{t=0} = 0$. It can be seen that the buckling of the system is sensitive to the initial values, and the attraction domain of A is broader than that of attractor B. Meanwhile, the symmetry mentioned earlier is broken due to damping.

The occurrence of buckling for this system depends on not only the system parameters, but also on the initial conditions associated with $\bar{k}$ (the load level defined before), and snap-through will occur in this case.

Case 3, Damped System with One Equilibrium Position

By the change of only the parameter G_{xy} to 2×10^{10} N/m^2, one obtains $\Delta < 0$. This means that the system has one equilibrium position $w_1 = 0$ only; thus, no buckling can occur. The real parts of the corresponding eigenvalues are negative, that is, the equilibrium position is locally stable. After impact, the system will approach this position again. Figure 4 shows the corresponding phase portrait. It can be seen that for this system different initial values will yield the same equilibrium postion.

Case 4, Benchmark: Response of an Isotropic Shell Under Uniform Load

In this case, the external force q^* is kept constant with respect to time, whereas both inertia and damping are neglected to allow for a comparison with results of an example in Ref. 14.

In Ref. 14, the nonlinear behavior of a rectangular doubly curved isotropic shallow shell has been studied, subjected to uniform loading and simply supported boundary conditions. In the solution, only the first mode pair has been considered.

Hence, let $n = m = 1$, $a = b$, $E_1 = E_2 = E$, $\gamma_{12} = \gamma_{21} = \gamma$, $G_{12} = E/2(1+\gamma)$, $w_{nm} = h\xi$, $\rho h q^* = -q^{**}(Eh^4/a^4)$, and $k = k_x^* + k_y^* = (a^2/h)(k_x + k_y)$. Now, the equation of motion of the system [Eq. (12)] becomes

$$D_{70}\xi^3 + D_{71}\xi^2 + D_{72}\xi - q^{**} = 0 \tag{27}$$

where

$$D_{70} = \pi^6/128, \qquad D_{71} = -(5k\pi^2/24)$$

$$D_{72} = \frac{1}{16}\left[\frac{\pi^6}{3(1-\gamma^2)} + \frac{k^2\pi^2}{4}\right]$$

For the analysis of stability of the equilibrium positions a point mapping is constructed by rewriting Eq. (27) as follows for the case $D_{72} \neq 0$:

$$\xi \mapsto G(\xi), \qquad \xi = D_{73}\xi^3 + D_{74}\xi^2 + D_{75}q^{**} \tag{28}$$

where

$$D_{73} = -(D_{70}/D_{72}), \qquad D_{74} = -(D_{71}/D_{72}), \qquad D_{75} = 1/D_{72}$$

Let ξ_0 be the fixed point; then its stability can be analyzed from

$$\lambda = \left.\frac{DG}{D\xi}\right|_{\xi=\xi_0} = 3D_{73}(\xi_0)^2 + 2D_{74}\xi_0 \tag{29}$$

There are four possibilities of the stability of the fixed point: First, if $|\lambda| > 1$, the fixed point is unstable. Second, if $|\lambda| < 1$, the fixed point

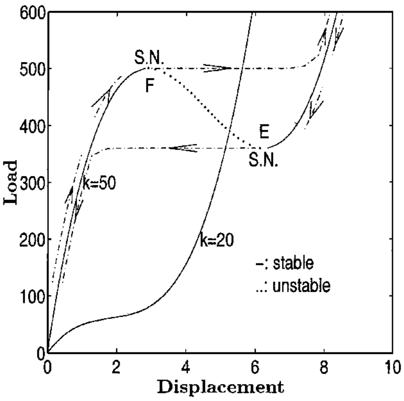

Fig. 5 Response vs load.

will be stable. Third, if $\lambda = 1$, a saddle node, transcritical or pitchfork bifurcations may occur. Fourth, if $\lambda = -1$, a period 2 mapping may occur.

To compare these results with known results from the literature,[14] let $\gamma = 0.3$.

For a first comparison, let $k = 20$. The response governed by Eq. (27) vs q^{**} is shown in Fig. 5. It is clear that there exists only one curve and no jumping. These results are the same as those given in Ref. 14.

For a second comparison, let $k = 50$. Now, the system has two saddle-node bifurcations, and the corresponding response is shown in Fig. 5. It is clear that there is only one solution in the range $q^{**} \in [0, 360.52]$. At point E(6.26, 360.52), the system has a saddle-node bifurcation; the corresponding behavior is jumping, as shown by arrows. In the range $q^{**} \in [360.52, 499.45]$, the system has three solutions, the middle one of which is unstable, while the others are stable. At point F(2.904, 499.45), another saddle-node bifurcation occurs; its behavior is the same as the first one at point E. The system has only one solution in the range $q^{**} \in (360.52, 600]$, which means that, for these parameters, the system has one equilibrium position only. By increasing q^{**} slowly from zero, initially solutions are attracted to the left branch of the solution curve and stay on it until jumping (at point F) to the right branch. On the other hand, if q^{**} is decreased slowly from values larger than that at point F, the solution will remain on the right branch before jumping (at point F) to the left branch. In other words, the response is associated with initial values. This phenomenon is defined as hysteresis. Indeed, jumping and hysteresis are typical phenomena in nonlinear dynamic systems.

Notice that in Ref. 14 also three solutions are given in the range $q^{**} \in [360.52, 499.45]$, but no stability analysis based on bifurcation theory. For the doubly curved panels, a bifurcation buckling may exist before reaching the maximum collapse load,[15] that is, the existence of the bifurcation lying between O and F has not been investigated further in Ref. 14.

Conclusions

It can be concluded that the methodology presented is efficient for the analysis of the stability of doubly curved orthotropic shallow shells under impact, from the viewpoint of nonlinear dynamics.

In particular, for the doubly curved orthotropic shallow shells under impact studied, two centers and one saddle will exist if damping is ignored, that is, for the associated Hamiltonian system. On the other hand, the two centers become two focuses and the saddle remains if damping is considered. Moreover, if the condition for buckling, associated with a specific set of system parameters, is satisfied, the occurrence of buckling will depend on the initial conditions (initial displacement and impact load).

In addition, for comparison with existing results from literature, the stability behavior has also been investigated by neglecting the influences of inertia and damping (resulting in quasi-static behavior). Point mapping is used to analyze the stability of the equilibrium positions. The results show that saddle-node bifurcations occur.

The influence of higher modes on the response to the impact load and the interaction between them are currently under investigation.

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